An Iterative Measured Equation Technique for Electromagnetic Problems

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***Abstract*—An iterative measured equation technique (IMET) is presented for a numerical solution of electromagnetic problems. This technique is an extension and improvement of the method of measured equation of invariance (MEI). In this technique, an iterative scheme is designed in such a way that a new set of metrons used to generate the measured equations is formed in each iteration based on the solution of the previous iteration. The new metrons are more meaningful in that they converge to the physical quantity of interest such as the surface current density for electrodynamic problems and the surface charge density for electrostatic problems. The IMET offers several advantages over the MEI method because it requires only two mesh layers, resulting in a significant reduction in the memory requirement and computing time. More importantly, it provides a means for a systematic improvement of the accuracy of solution. The IMET is applied successfully to two-dimensional (2-D) electrodynamic and three-dimensional (3-D) electrostatic problems. Numerical results show that the technique is highly accurate and the iterative process converges very quickly, usually within two iterations.**

***Index Terms*—Absorbing boundary condition, electromagnetic scattering, iterative algorithm.**

I. Introduction

When solving open-region electromagnetic problems using the finite-difference or finite-element method, the infinite region exterior to the object must be truncated with an artificial boundary to limit the number of unknowns. Consequently, a boundary condition must be introduced at this artificial boundary for a unique solution. Two classes of such boundary conditions have been developed in the past. The first class of boundary conditions is derived from the boundary integral equations (BIE's) involving Green's functions. These boundary conditions are exact and can be applied directly at the surface of the object. However, because of the Green's function, the numerical system corresponding to such boundary conditions is a dense matrix, which is expensive to solve and store. The second class of boundary conditions is derived from the differential wave equations and are called absorbing boundary conditions (ABC's). These boundary conditions maintain the sparsity of the system matrix. However, they do not lead to exact solutions because of the approximations introduced in their derivation.

Recognizing the shortcomings of both BIE's and ABC's, Mei *et al.* have introduced the method of measured equation of invariance (MEI) [1]. This method derives localized boundary conditions from global BIE's and, as a result, is more accurate than ABC's. It has been applied successfully to a number of problems, such as the scattering analysis [2]–[4], microstrip discontinuity characterization [5], and parameter extraction in VLSI circuits [6], [7]. In the MEI method, the accuracy of the solution is mainly controlled by the number of mesh layers surrounding the object, the number of points or nodes at which the measured equations are calculated, and the number of metrons used to generate the measured equations. Generally speaking, the solution accuracy improves as these numbers increase. However, it is difficult to estimate and improve the accuracy and know whether the solution has converged to the true solution without performing new computations using a different number of mesh layers, MEI nodes, or metrons. Furthermore, it is also difficult to utilize the result from the previous calculations in the new computation.

In this paper, we present an iterative technique to alleviate the limitations of the MEI method described above. In this technique, the accuracy of the solution is improved through an iterative process without increasing the number of mesh layers, MEI nodes, and metrons. This iterative scheme is designed in such a way that a new set of metrons used to generate the measured equations is formed in each iteration based on the solution of the previous iteration. The new metrons are more meaningful in that they converge to the physical quantity of interest such as the surface current density for electrodynamic problems and the surface charge density for electrostatic problems. This method is termed as the iterative measured equation technique (IMET) and its most important advantage is that it provides a means for a systematic improvement of the accuracy of the solution. Because of this, the number of mesh layers used in the IMET can be reduced to two, resulting in a significant reduction in the memory requirement and computing time. To a certain extent, the idea of the IMET is similar to that of the adaptive ABC developed for two-dimensional (2-D) problems by Yi and Cendes [8] and for three-dimensional (3-D) problems developed by Jin and Lu [9]. In this paper, we apply the IMET to 2-D electrodynamic and 3-D electrostatic problems and demonstrate through numerical results that the technique

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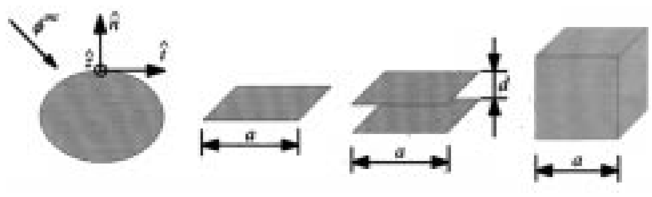

Fig. 1. Configuration of the problems. (a) Cylinder. (b) Square plate. (c) Two parallel plates. (d) Cube.

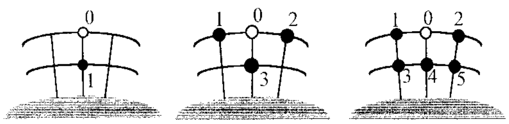

Fig. 2. Arrangement of MEI nodes for 2-D problems. (a) Two-point node. (b) Four-point node. (c) Six-point node.

is highly accurate and the iterative process converges very quickly, usually within two iterations.

II. Formulation

This section describes the basic principle of the IMET.

A. The Problems

For simplicity, we consider only 2-D electrodynamic and 3-D electrostatic problems. However, the same idea can be applied to other problems without any conceptual difficulty.

The 2-D electrodynamic problem is illustrated in Fig. 1(a), where a TM polarized plane wave is scattered by an infinitely long conducting cylinder. We use ϕ to represent the scattered-field component E_z. Hence, ϕ satisfies the Helmholtz equation

$$(\nabla^2 + k_0^2)\phi = 0 \tag{1}$$

where ∇^2 is the Laplace operator in the 2-D coordinate system and k_0 denotes the free-space wavenumber. The problem is to find ϕ for a given incident field ϕ^{inc}.

The 3-D electrostatic problems to be considered are depicted in Fig. 1(b)–(d). The problem is to find the potential and charge distributions on a square plate, two-plate capacitor, and cube for a given voltage. The electric potential, denoted as ϕ, satisfies the Laplace equation

$$\nabla^2 \phi(x, y, z) = 0 \tag{2}$$

where ∇^2 is the 3-D Laplace operator.

B. Discretization

Similar to the MEI method, a discretization is made outside the object surface. However, only two mesh layers are needed in the IMET. The mesh layers for the 2-D problem are shown in Fig. 2, along with the arrangement of the MEI nodes. The meshes for the 3-D problems are formed by two boxes surrounding the object, as illustrated in Fig. 3(a), for the case of a square plate. Also shown in Fig. 3 is the arrangement of the MEI nodes for a surface, edge, and corner node.

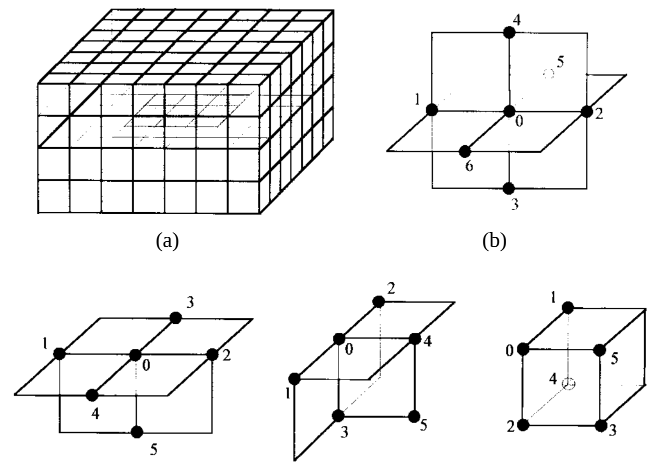

Fig. 3. Discretization schematics and arrangements of MEI nodes for 3-D problems. (a) Rectangular plate surrounded by boxes. (b) Seven-point FD scheme. (c) MEI nodes in a surface. (d) MEI nodes on an edge. (e) MEI nodes at a corner.

The advantages of using the two-layer discretization are obvious. First, it saves memory since the number of unknowns is significantly reduced compared to a multilayer discretization required in the MEI method. Second, it reduces the dispersion error in the finite-difference solution since this error is proportional to the size of the discretization region. Third, it simplifies the task of mesh generation; thus, making the programming much easier.

C. IMET

Since the electrostatic problem is a limiting case of an electrodynamic problem when the frequency approaches zero, we describe the basic principle of the IMET only for the electrodynamic problem whose differential equation is given in (1). The five-point finite-difference discretization of (1) yields the equation

$$\sum_{k=0}^{4} a_k \phi_k = 0. \tag{3}$$

Applying this equation to the nodes on the interior layer, we obtain the matrix equation

$$[A][\phi] = [b] \tag{4}$$

where $[\phi] = [\phi_1, \phi_2, \cdots, \phi_p, \phi_{p+1}, \cdots, \phi_{p+q}]^T$, in which p denotes the number of unknowns on the interior layer and q denotes the number of unknowns on the exterior layer. Also, $[A]$ is a sparse matrix having a dimension of $p \times (p+q)$, and $[b]$ is a column vector related to the incident field.

To obtain the remaining q equations, we apply the MEI method to a node on the exterior layer, yielding the equation

$$\sum_{k=0}^{n-1} b_k \phi_k = 0 \tag{5}$$

where b_k are the MEI coefficients to be determined, and n denotes the number of points in the MEI scheme, which can

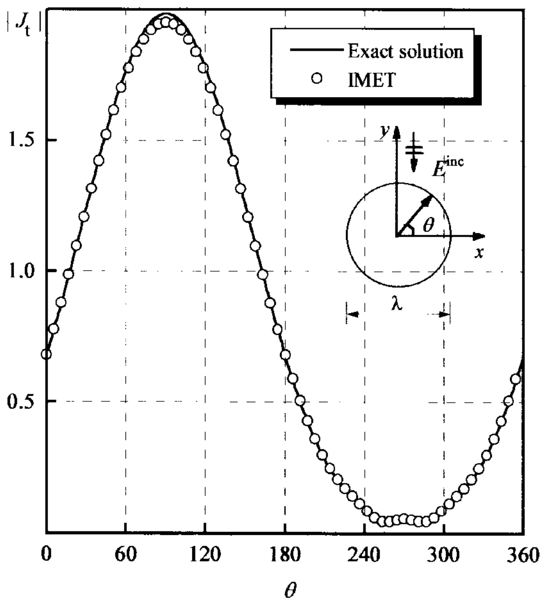

Fig. 4. Surface current on a circular conducting cylinder having a radius of 0.5λ.

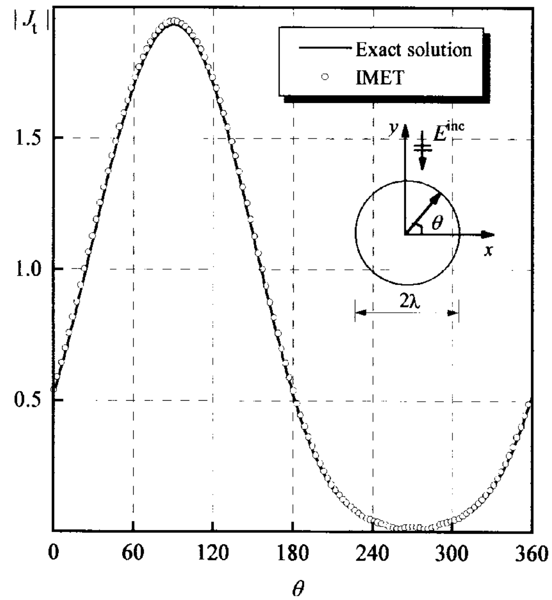

Fig. 5. Surface current on a circular conducting cylinder having a radius of 1.0λ.

be two, four, and six, as shown in Fig. 2. Applying (5) to the nodes on the exterior layer, we obtain another matrix equation

$$[B][\phi] = 0 \tag{6}$$

where $[B]$ is a sparse matrix having a dimension of $q\times(p+q)$. Combining (4) and (6), we obtain the complete system of equations

$$\begin{bmatrix} A \\ B \end{bmatrix}[\phi] = \begin{bmatrix} b \\ 0 \end{bmatrix} \tag{7}$$

which can be solved for $[\phi]$.

It remains to determine the coefficients b_k in (5). For this, we introduce M different metrons $\psi_j(\overrightarrow{r}')$ assumed on the object surface. Using these metrons as the current density to measure the scattered field on the nodes, we have

$$\phi_{ij} = \int_\Gamma \psi_j(\overrightarrow{r}')G(\overrightarrow{r}_i, \overrightarrow{r}')\,dl', \qquad j = 1, 2, \cdots, M \tag{8}$$

where $G(\overrightarrow{r}_i, \overrightarrow{r}')$ is the Green's function and Γ denotes the surface of the object. Using (8) and following the same procedure as described in [1], we can determine the coefficients b_k in (5) and, thus, the coefficient matrix $[B]$ in (6). Since $[B]$ is determined from the metrons, we can symbolically write

$$[B] = \mathcal{L}(\overline{\psi}) \tag{9}$$

where $\overline{\psi}$ is a column vector denoting the metrons used.

The procedure describe above is that of the standard MEI method. However, since we use only two mesh layers here, the matrix relation (6) is not accurate enough for a good solution. In this case, we can calculate the current density $\overrightarrow{J} = \hat{n}\times\overrightarrow{H}$ based on the solution of $[\phi]$. This current is then used to replace the first metron and the new set of metrons can be written symbolically as

$$\overline{\psi} = \mathcal{H}([\phi]). \tag{10}$$

Using this new set of metrons, we can find a new $[B]$, which yields a new solution by solving (7) again. The process can be repeated until a convergence is achieved. This iterative process can be expressed as

$$[\phi]^{(k+1)} = \begin{bmatrix} A \\ \mathcal{L}(\overline{\psi}^{(k+1)}) \end{bmatrix}^{-1} \begin{bmatrix} b \\ 0 \end{bmatrix} \tag{11}$$

where

$$\overline{\psi}^{(k+1)} = \mathcal{H}\Big([\phi]^{(k)}\Big). \tag{12}$$

Note that in (12), the new set of metrons is obtained by replacing the $(k+1)$th existing metron with the new metron obtained from $[\phi]^{(k)}$. If $k+1 > M$, the $(k+1\text{-}M)$th existing metron is replaced with the new metron.

To check the convergence of solution, an error criterion is defined as

$$e = \frac{\|\overline{J}^{(k+1)} - \overline{J}^{(k)}\|}{\|\overline{J}^{(k+1)}\|} \tag{13}$$

where $\|*\|$ stands for the norm of vector. We note that as the iteration proceeds, all the metrons become the same. As a result, the matrix equation to determine the coefficients in (5) would become singular. However, this does not happen because the iteration is terminated when the metrons converge to the same value.

III. Numerical Results

This section presents some numerical examples to demonstrate the performance and the accuracy of the IMET.

A. Scattered-Field Computation

We first apply the IMET to calculate the surface electric current on a circular conducting cylinder illuminated by a TM plane wave. The numerical results are given in Figs. 4–6 for

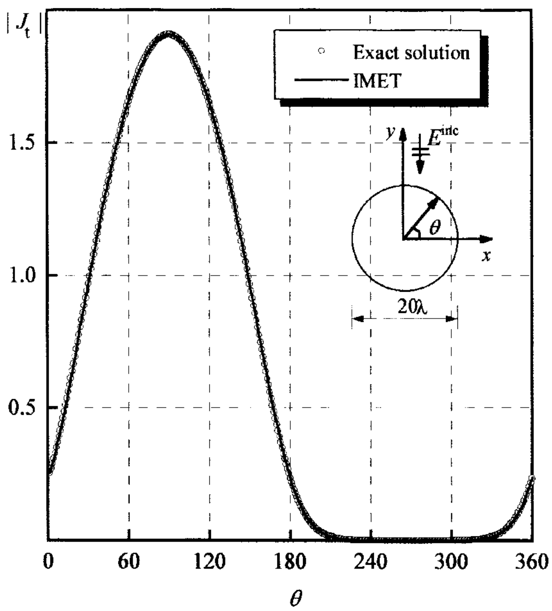

Fig. 6. Surface current on a circular conducting cylinder having a radius of 10λ.

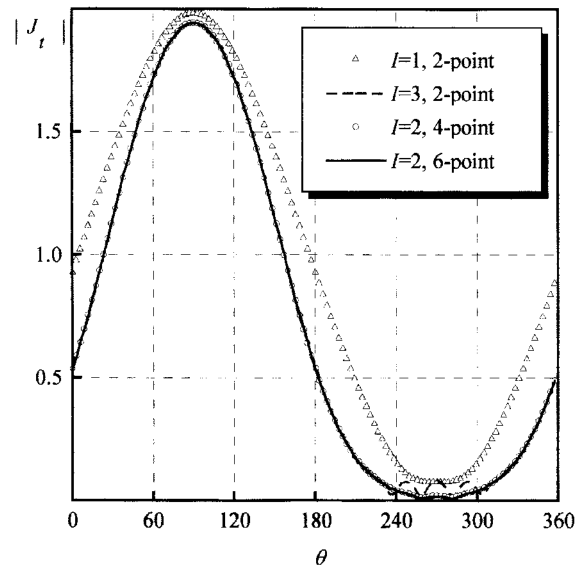

Fig. 7. Surface current on a circular conducting cylinder having a radius of computed 1.0λ using different number of points and iterations.

TABLE I
COMPARISON OF CPU TIME

	IMET				MEI	MoM
Iterations	1	2	3	4		
Inversion	1.6 s	3.2 s	4.8 s	6.4 s	14.4 s	
Total	2.7 s	5.1 s	7.4 s	9.8 s	16.9 s	57.4 s

cylinders having a radius of 0.5λ, 1.0λ, and 10λ, respectively. As can be seen, the IMET has a high accuracy and is valid for a wide range of dimension. For the same accuracy, the MEI method requires more mesh layers and more metrons. For example, in the case of the cylinder having a radius of 1.0λ, the MEI method requires six mesh layers and eight metrons to obtain an accuracy that can be achieved by the IMET with only two mesh layers, three metrons, and two iterations. As a result, the IMET saves memory by a factor of three and computing time by a factor of 3.3 since it solves a smaller banded matrix with a smaller bandwidth. For this problem, it is observed that the numerical results at and after the third iterations are almost the same as those of the second iteration, which indicates a fast convergence of the IMET. The central processing unit (CPU) time on a PC586/100 needed for this problem is given in Table I and is compared to those required for the MEI method and the method of moments (MoM).

The accuracy of different MEI schemes, namely the two-, four-, and six-point schemes, is compared in Fig. 7, where I denotes the number of iterations. The reason to employ the two-point scheme is that it uses fewer metrons and less memory and it avoids the matrix inversion in the determination of the coefficients in (6). However, the results show that this scheme is less accurate even after three iterations because it does not model the variation along the direction tangential to the object's surface; therefore, the two-point scheme is not recommended. On the other hand, although the four- and six-point schemes require slightly more memory and CPU time, their solutions have a good accuracy and converges very quickly.

We then apply IMET to a $1\lambda \times 1\lambda$ square cylinder, with the result shown in Fig. 8. Clearly, the agreement between the IMET and MoM results is better than that between the MEI and MoM results. Our numerical experiments reveal that an increase in the number of metrons does not always lead to an effective improvement in the accuracy of the solution. On the contrary, it can slow down the convergence of the IMET. This is demonstrated by the results obtained using three and eight metrons after two iterations, shown in Fig. 9. The reason is that the coefficients in (6) are determined by the least-squares method with the nodal values measured by different metrons. The weight of the metron obtained using the solved current is lessened when the number of metrons is increased. This weight can only be increased with the number of iterations. Thus, with only two iterations, the IMET using eight metrons can give a less accurate result than the IMET with three or five metrons.

Finally, we apply the IMET to a concave conducting cylinder and the result is shown in Fig. 10. Again, the agreement between the IMET and MoM results is better than that between the MEI and MoM results.

B. Capacitance Calculation

When the frequency approaches zero, an electrodynamic problem becomes an electrostatic problem. The surface charge density on the object is then employed as the metron to measure the electrical potential in the MEI nodes. We first calculate the capacitance of a unit square plate using two boxes to enclose the plate. The capacitance calculated by the MEI method is $C = 37.748$ pF, and that by the IMET is $C = 39.945$ pF after one iteration. The approximate value given in [10] is $C = 39.94$ pF and that calculated by the MoM is $C = 39.5$ pF.

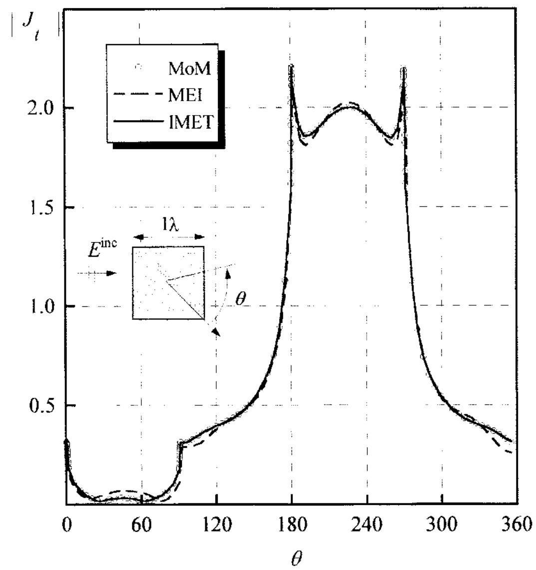

Fig. 8. Surface current on a square conducting cylinder having side length of 1.0λ.

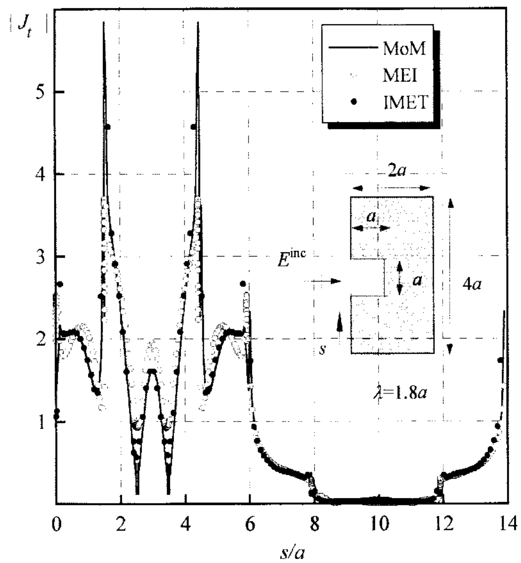

Fig. 10. Surface current on a concave conducting cylinder.

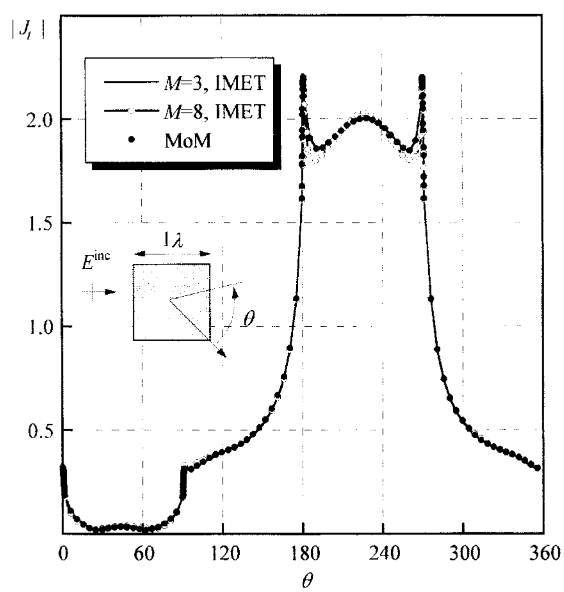

Fig. 9. Surface current on a square conducting cylinder having a side length of 1.0λ computed using different number of metrons.

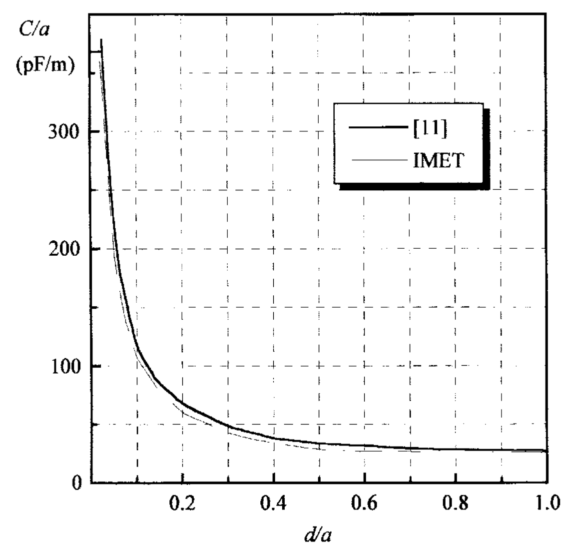

Fig. 11. Capacitance of two parallel plates versus their separation.

We then consider a parallel-plate capacitor of unit square. The image theory is employed to determine the Green's function in this case since the symmetric plane between the two plates can be considered as the ground if the two plates are imposed with opposite excitations. The calculated capacitance is plotted in Fig. 11 as a function of the distance between the plates and compared to that in [11]. The value calculated in the second iteration is almost the same as that calculated in the first iteration. As expected, the value of capacitance approaches half the value of one plate alone in space, namely 20 pF, as $d/a \to \infty$.

Finally, we consider a more general 3-D object, which is the unit cube in Fig. 1(d). By symmetry, the discretization is made for one eighth of the cube. The MEI method gives the value of capacitance $C = 61.466$ pF. With the IMET, the value is $C = 73.286$ pF after the first iteration and $C = 72.588$ pF after the second iteration. The available result is found in [12] with $C = 72.322$ pF. The MEI method takes quite a few mesh layers around the cube to obtain the approximate solution, whereas IMET uses only two mesh layers with one iteration. The charge density on one surface of the cube is plotted in Fig. 12. The charge density has a maximum value of 62.52 C/m^2 at the cube's corners and a minimum value of 6.68 C/m^2 at the center of the surface.

IV. Conclusion

In this paper, an IMET is presented for a numerical solution of electromagnetic problems. In this technique, an iterative scheme is designed in such a way that a new set of metrons used to generate the measured equations is formed in each iteration based on the solution of the previous iteration. The new metrons are more meaningful in that they converge to

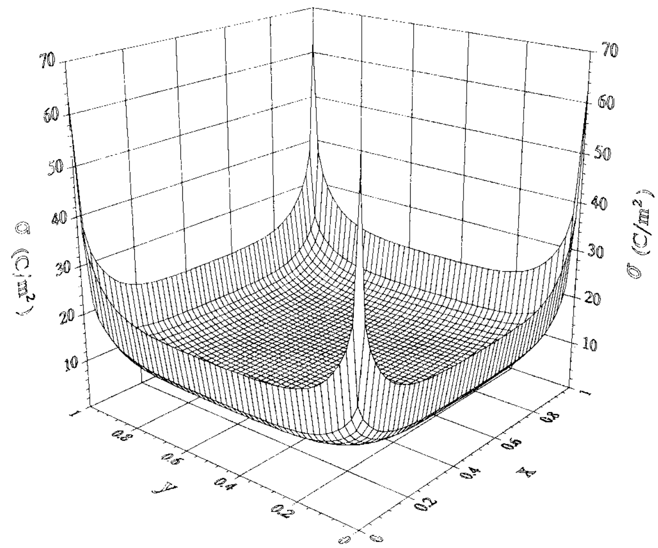

Fig. 12. Charge density on one surface of a cube.

the physical quantity of interest such as the surface current density for electrodynamic problems and the surface charge density for electrostatic problems. The IMET requires only two mesh layers, resulting in a significant reduction in the memory requirement and computing time. More importantly, it provides a means for a systematic improvement of the accuracy of solution. The IMET is applied successfully to 2-D electrodynamic and 3-D electrostatic problems. Numerical results show that the technique is highly accurate and the iterative process converges very quickly, usually within two iterations.

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